IB Mathematics: Applications and Interpretations HL
Internal Assessment

THE COASTLINE PARADOX

Investigating The Coastline Paradox In Relation To The Moroccan Territory

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Contents

1 Introduction **1**
- 1.1 Rationale . . . 1
- 1.2 Aim . . . 1

2 The Coastline Paradox **1**
- 2.1 Background Information . . . 1
- 2.2 Dimensions . . . 2
  - 2.2.1 Fractal Dimension of the Koch Snowflake . . . 2

3 Mathematical Methods to Determine Fractal Dimensions **3**
- 3.1 The Hausdorff Method . . . 3
  - 3.1.1 Determining the Fractal Dimension . . . 6
- 3.2 The Minkowski Bouligand Method . . . 6
  - 3.2.1 Determining the Fractal Dimension . . . 9
- 3.3 The Relationship Between the Surface Area & Perimeter of a Fractal Structure . . . 9
  - 3.3.1 Determining the Surface Area of Britain via Green's Theorem . . . 10

4 Determining the Fractal Dimension of Morocco **12**
- 4.1 Percentage Error Analysis . . . 12
- 4.2 Hausdorff Method to Determine Fractal Dimension of Morocco . . . 12

5 Discussion **15**
- 5.1 Conclusion & Evaluation . . . 15

6 Works Cited **16**

1 Introduction

1.1 Rationale

Although our lips were tinged with blue and our teeth gently chattering, my father and I persisted on our stroll on the Agadir beach. As the cold crept under our clothes, we continued to dip our feet in the ocean tide. Out of the blue, my father, a naturally inquisitive individual, asked me "Where does this beach end?" I paused for a moment, unable to answer his question. I realized that what one may consider being the beach depended on the strength of the ocean tide on a given day; if the tide happens to be quite strong one day, then less of the beach is present (i.e. it's covered by the tide) and vice versa. To answer my father's question, I began to research which areas of the beach would be considered by the Moroccan government when establishing the length of its coastline. I found two conflicting answers. The CIA reported that the Moroccan coastline was 1,835 kilometers in length [1] while another source stated that it was 1,200 kilometers [2]. Did these sources take the measurements on different days with different tide strengths? Both sources were quite recent, so it wasn't as if the Moroccan coastline had grown in the last couple of years. Why was I obtaining subjective answers to an objective question? Upon further research, I found out that I could explain this dilemma through the Coastline Paradox. In essence, if I was to measure the coastline of Morocco using a 100 centimeter stick versus a 100 meter stick, I would obtain a greater value for the measurement of the coastline with the 100 centimeter stick since it allows for the measurement of areas that the larger ruler couldn't reach. Thus, theoretically speaking, if I was to measure the coastline of Morocco via molecules, I would obtain a value so large that it wouldn't have any applicable meaning. Diverging from my father's question, I began to ask my own. How can I most meaningfully, yet accurately, measure the coastline of Morocco?

1.2 Aim

Ultimately, this paper seeks to identify the most accurate and meaningful technique of measuring the coastline of Morocco out of the following three mathematical methods: the Hausdorff method, the Minkowski Bouligand method, and via the manipulation of the relationship between the surface area and perimeter of a fractal structure.

2 The Coastline Paradox

2.1 Background Information

In his paper titled *How long is the coast of Britain?*, Benoit Mandelbrot explored the idea of fractals and how they relate to Great Britain's coast. He explains what is now known as the Coastline Paradox in the following expert from his paper:

> Seacoast shapes are examples of highly involved curves with the property that, in a statistical sense, each portion can be considered a reduced-scale image of the whole. This property will be referred to as "statistical self-similarity." The concept of "length" is usually meaningless for geographical curves. They can be considered superpositions of features of widely scattered characteristic sizes; as even finer features are taken into account, the total measured length increases, and there is usually no clear-cut gap or crossover, between the realm of geography and details with which geography need not be concerned [3].

2.2 Dimensions

Before embarking on this investigation, one must first understand the phenomena of dimension. In short, a dimension can be understood as a "measurement of length in one direction" [4]. Similarly, Mandelbrot defined dimension as the following: an n-dimensional figure is a figure that when scaled by a factor of $\frac{1}{x}$ results in x^n copies of itself [3]. He represented this definition through the following formula [3]:

$$(\frac{1}{r})^D = N \tag{1}$$

r = Scale factor
D = Dimension
N = Number of copies

Through logarithmic laws, one can manipulate (1) in order to solve for D:

$$(\frac{1}{r})^D = N$$

$$\log{(\frac{1}{r})}^D = \log N \tag{2}$$

$$\log{(\frac{1}{r})} * D = \log N \tag{3}$$

$$D = \frac{\log N}{\log(\frac{1}{r})} \tag{4}$$

2.2.1 Fractal Dimension of the Koch Snowflake

There is a common misconception that dimensions can only exist in discrete values. For instance, a line would have one dimension, a square would have two dimensions, and a cube would have three dimensions. However, the latter misconception is not true as there are structures that can have fractal dimensions. The famous Koch Snowflake is an example of such a structure. The Koch Snowflake is said to have an infinite perimeter and a finite area, thus possessing fractal properties similar to those of coastlines. Hence, being able to understand the Koch Snowflake's fractal dimension will allow one to understand the complexity of its edges.

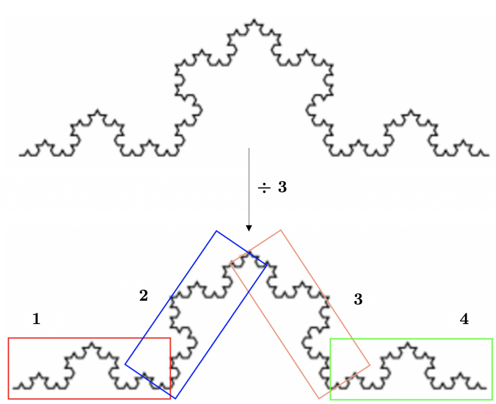

Figure 1: Number of copies (N) created by using a scale factor (r) of $\frac{1}{3}$ on a section of the Koch Snowflake

As shown in **Figure 1**, applying a scale factor (r) of $\frac{1}{3}$ onto a section of the Koch Snowflake has resulted in the creation of four copies of the structure (N). Thus, through (4), one can calculate the value of the Koch Snowflake's fractal dimension:

$$D = \frac{\log N}{\log(\frac{1}{r})}$$

$$D = \frac{\log 4}{\log(\frac{1}{\frac{1}{3}})} \tag{5}$$

$$D = 1.26 \tag{6}$$

3 Mathematical Methods to Determine Fractal Dimensions

As there are several methods that one can use to determine the fractal dimension of a country, it is paramount that one must establish which is the most accurate and meaningful. This section will focus on determining the fractal dimension of Britain through three different methods: the Hausdorff method, the Minkowski Bouligand Method, and via the manipulation of the relationship between the surface area and perimeter of a fractal structure. The fractal dimension value that is obtained from each method will be compared to the literature value of Britain's fractal dimension, as proposed by Mandelbrot ($D \approx 1.25$) [3]. The method with the lowest percentage error will then be utilized to determine the fractal dimension of Morocco.

3.1 The Hausdorff Method

First introduced by Felix Hausdorff, the Hausdorff method allows one to determine the fractal dimension of a structure with crude edges (i.e. a coastline) [5]. This method utilizes an arbitrarily sized ruler (G) in order to approximate the perimeter of the territory of the country [5]. The smaller the size of the ruler, the greater the perimeter of the territory, as a smaller ruler allows one to measure the finer details of the territory and coastline. Mandelbrot shows this relationship in the following equation [3]:

$$L = G^{(1-D)}M \tag{7}$$

L = Length of coastline
G = Length of ruler
D = Dimension
M = Proportionality constant

Through logarithmic laws, (7) can be converted into a $y = mx + b$ equation, thus allowing for the isolation of the dimension variable, D:

$$L = G^{(1-D)}M$$

$$\log L = \log G^{(1-D)}M \tag{8}$$

$$\log L = \log G^{(1-D)} + \log M \tag{9}$$

$$\log L = (1 - D) \log G + \log M \tag{10}$$

$$\therefore$$

$$y = \log L \tag{11}$$

$$m = 1 - D \tag{12}$$

$$x = \log G \tag{13}$$

$$b = \log M \tag{14}$$

To visualize the Hausdorff method, I used Photoshop to apply different sized rulers around the coast of Britain, as shown in the following figures.

Figure 2: Modeling British coastline with 200 km ruler. The original image of the map was taken from Bella Caledonia [6]

	Length of Ruler (G) (km)	$\log G$	Number of Rulers (n)	Length of Coastline (L)	$\log L$
Values	200	2.30	15	3000	3.47

Table 1: Properties of **Figure 2** needed to determine fractal dimension (D)

Figure 3: Modeling British coastline with 100 km ruler

	Length of Ruler (G) (km)	$\log G$	Number of Rulers (n)	Length of Coastline (L)	$\log L$
Values	100	2.00	33	3300	3.52

Table 2: Properties of **Figure 3** needed to determine fractal dimension (D)

Figure 4: Modeling British coastline with 50 km ruler

	Length of Ruler (G) (km)	$\log G$	Number of Rulers (n)	Length of Coastline (L)	$\log L$
Values	50	1.70	85	4250	3.63

Table 3: Properties of **Figure 3** needed to determine fractal dimension (D)

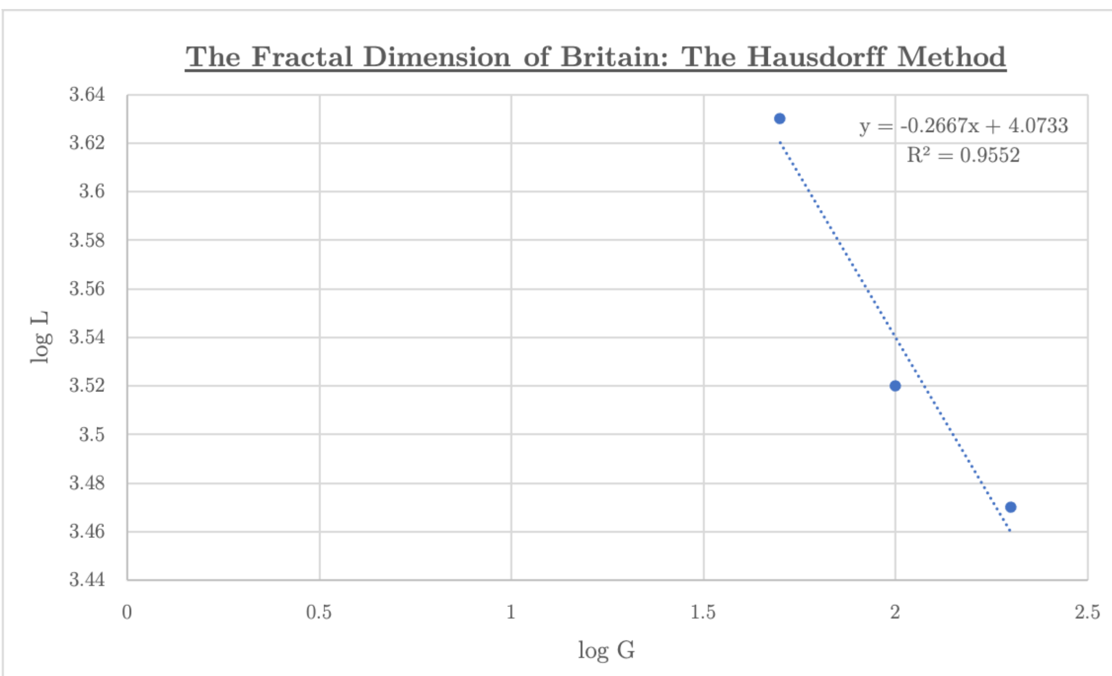

Figure 5: Plot of $\log G$ and $\log L$ from **Tables 1 - 3** to determine fractal dimension (D) of Britain

3.1.1 Determining the Fractal Dimension

As shown in (12), the fractal dimension can be determined from the slope of **Figure 5**:

$$1 - D = m$$

$$D = 1 - m$$

$$D = 1 - (-0.2667)$$

$$D = 1.2667$$

Next, the percentage error between the fractal dimension determined via the Hausdorff method and Mandelbrot's proposed value for the fractal dimension of Britain (i.e. the literature value) will be determined:

$$\delta = \left| \frac{(V_A) - (V_E)}{V_E} \right| 100$$

$$\delta = \left| \frac{(1.2667) - (1.25)}{1.25} \right| 100$$

$$\delta = 1.336\% \tag{15}$$

δ = Percentage error
V_A = Actual value observed
V_E = Expected value

3.2 The Minkowski Bouligand Method

The Minkowski Bouligand method, also known as the box-counting method, is a method where different sized grids are superimposed upon the chosen structure (i.e. the map of Britain). The method states that one will count the number of boxes that touch the coastline of the chosen structure. Similar to the Hausdorff method, it is expected that as the grid gets smaller, the number of boxes that are in contact with the coastline of the chosen structure will increase. This relationship is shown via the following equation [7]:

$$N = S^D C \tag{16}$$

N = Number of boxes
S = Scale factor
D = Dimension
C = Proportionality constant

Again, through logarithmic laws, (16) can be converted into a $y = mx + b$ equation, thus allowing for the isolation of the dimension variable, D:

$$N = S^D C$$

$$\log N = D \log S + \log C \tag{17}$$

$$\therefore$$

$$y = \log N \tag{18}$$

$$m = D \tag{19}$$

$$x = \log S \tag{20}$$

$$b = \log C \tag{21}$$

To visualize the Minkowski Bouligand method, I used Photoshop to superimpose different sized grids onto the map of Britain in order to determine the fractal dimension (D), as shown in the following figures. Note that certain squares are colored orange to indicate that they contain a portion of the British coastline.

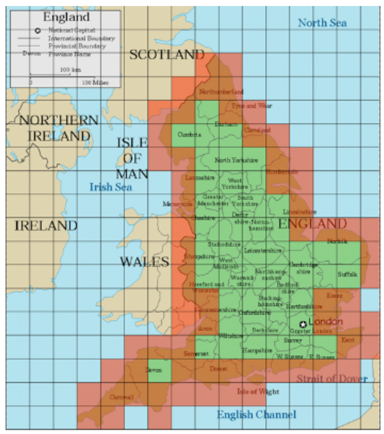

Figure 6: Modeling British coastline with a superimposed grid with scale factor (S) of 1

	Scale factor (S) (km)	$\log S$	Number of Boxes (N)	$\log N$
Values	1	0	65	1.81

Table 4: Properties of **Figure 6** needed to determine fractal dimension (D)

Figure 7: Modeling British coastline with a superimposed grid with scale factor (S) of $\frac{4}{3}$

	Scale factor (S) (km)	$\log S$	Number of Boxes (N)	$\log N$
Values	$\frac{4}{3}$	0.125	91	1.96

Table 5: Properties of **Figure 7** needed to determine fractal dimension (D)

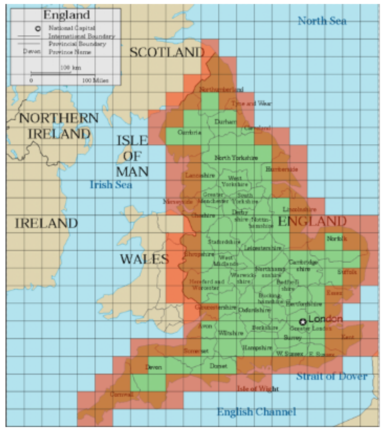

Figure 8: Modeling British coastline with a superimposed grid with scale factor (S) of 2

	Scale factor (S) (km)	$\log S$	Number of Boxes (N)	$\log N$
Values	2	0.301	146	2.16

Table 6: Properties of **Figure 8** needed to determine fractal dimension (D)

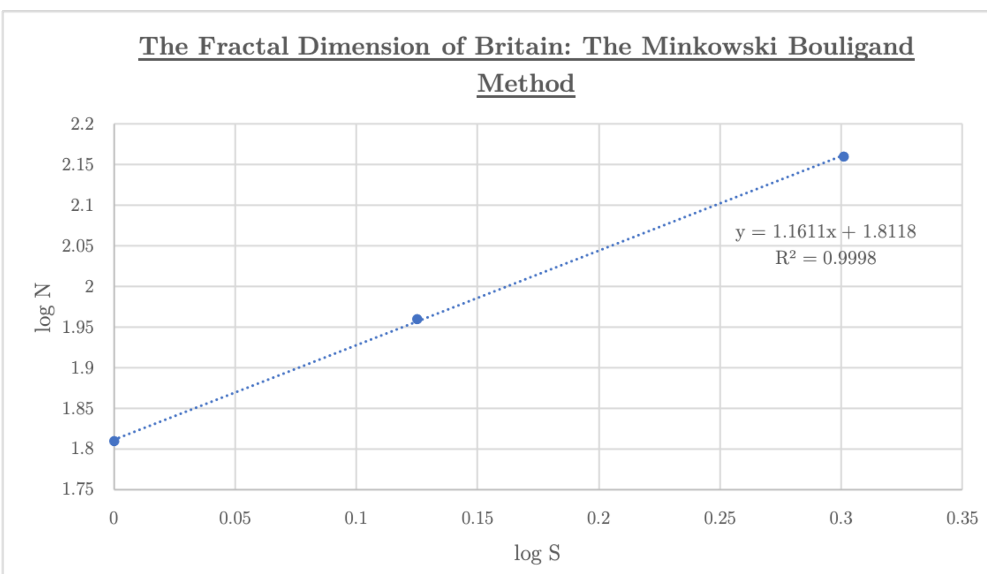

Figure 9: Plot of $\log S$ and $\log N$ from **Tables 4-7** to determine fractal dimension (D) of Britain

3.2.1 Determining the Fractal Dimension

As shown in (19), the fractal dimension can be determined from the slope of **Figure 9**:

$$D = m$$

$$D = 1.1611$$

Next, the percentage error between the fractal dimension determined via the Minkowski Bouligand method and Mandelbrot's proposed value for the fractal dimension of Britain will be determined:

$$\delta = \left|\frac{(V_A) - (V_E)}{V_E}\right| 100$$

$$\delta = \left|\frac{(1.1611) - (1.25)}{1.25}\right| 100$$

$$\delta = 7.112\% \tag{22}$$

3.3 The Relationship Between the Surface Area & Perimeter of a Fractal Structure

There is proportional relationship between surface area (A) and perimeter (P). However, to show this relationship, both variables must possess the same dimensions [8]. For instance, in order to compare an area to a line, the area must be to the power of $\frac{1}{2}$:

$$A^{\frac{1}{2}} \propto L^{\frac{1}{1}} \tag{23}$$

Knowing this need for both variables to possess the same dimensions (D), the following proportional relationship between surface area (A) and perimeter (P) can be derived [8][9]:

$$A^{\frac{1}{2}} \propto P^{\frac{1}{D}} \tag{24}$$

Feder and Shi et al. demonstrate a more nuanced version of (24), where the step-size variable G (i.e. length of ruler) and the constant c are present [9][10]:

$$\frac{P}{G} = c\left(\frac{A}{G^2}\right)^{\frac{D}{2}} \tag{25}$$

One can now utilize logarithmic laws to isolate the dimension variable (D):

$$\log\frac{P}{G} = \log c + \frac{D}{2}\left(\log\frac{A}{G^2}\right) \tag{26}$$

$$\begin{cases} \log\frac{P_1}{G_1} = \log c + \frac{D}{2}\left(\log\frac{A_1}{(G_1)^2}\right) \\ \log\frac{P_2}{G_2} = \log c + \frac{D}{2}\left(\log\frac{A_2}{(G_2)^2}\right) \end{cases} \tag{27}$$

$$\log\frac{P_2}{G_2} - \log\frac{P_1}{G_1} = \frac{D}{2}\left(\log\frac{A_2}{(G_2)^2} - \log\frac{A_1}{(G_1)^2}\right) \tag{28}$$

$$D = \frac{2\log\left(\frac{P_2 G_1}{P_1 G_2}\right)}{\log\left(\frac{A_2 (G_1)^2}{A_1 (G_2)^2}\right)} \tag{29}$$

To find the fractal dimension of Britain, I will utilize the two most precise perimeter measurements and step-sizes from **Table 2** and **Table 3**. However, the surface area of Britain remains to be determined. I recognize that the value for the surface area of Britain is readily available online ($130,395$ km^2), however, for the sake of mathematical rigor, I will determine this value through Green's theorem [12].

3.3.1 Determining the Surface Area of Britain via Green's Theorem

Green's theorem relates "the line integral of a two-dimensional vector field over a closed path in the plane and the double integral over the region it encloses" [13]. Thus, if the vector field $\mathbf{F} = \langle P, Q \rangle$ and the region **D** are "sufficently nice", and if **C** is the boundary of **D** (given that **C** is a closed curve), then the following is true [14]:

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{30}$$

Next, an expression for the region bounded by **D** can be found:

$$\oint_C P dx + Q dy = \iint_D (1 - (-1)) dA = 2 \iint_D dA = 2A \tag{31}$$

$$A = \frac{1}{2} \oint_C P dx + Q dy \tag{32}$$

We can modify Green's theorem to better model the borderline of Britain with the following discrete intervals:

$$dx = \Delta x = x_{i+1} - x_i \tag{33}$$

$$dy = \Delta y = y_{i+1} - y_i \tag{34}$$

By substituing (33) and (34) into (32), the following can be determined:

$$A = \frac{1}{2} \sum_{i=1}^{n} (x_i(y_{i+1} - y_i) - y_i(x_{i+1} - x_i)) \tag{35}$$

$$A = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - y_i x_{i+1}) \tag{36}$$

To use Green's theorem to determine the surface area of Britain, an outline containing the coordinates of **Figure 3** was created. Note that the outline between any two coordinates in **Figure 10** is approximately 100 km long:

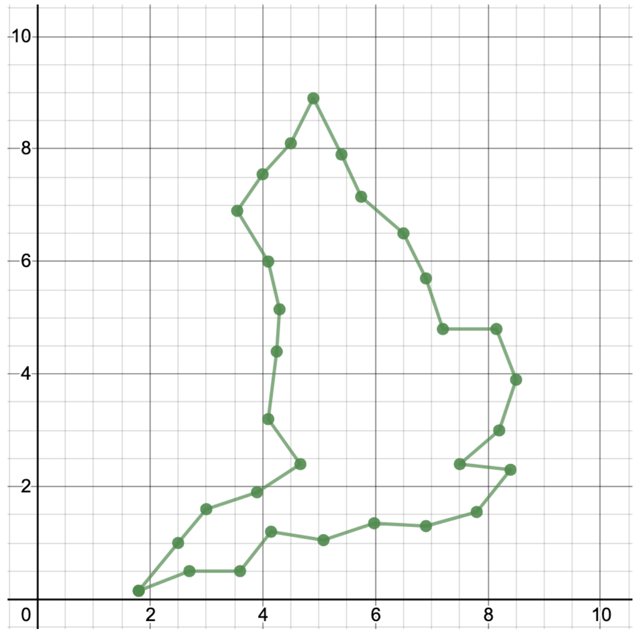

Figure 10: Outline containing the coordinates of **Figure 3**, created on Desmos

The surface area can now be found by using (36) and the coordinates from **Figure 10**:

x	y	Surface Area (unit^2)	x	y	Surface Area (unit^2)
1.80	0.15	$0.5((1.80 * 0.50) - (0.15 * 2.70)) = 0.25$	2.70	0.50	-0.23
3.60	0.50	1.12	4.15	1.20	-0.87
5.08	1.05	0.29	5.98	1.35	-0.77
6.90	1.30	0.28	7.80	1.55	2.46
8.40	2.30	1.46	7.50	2.40	1.41
8.20	3.00	3.24	8.50	3.90	4.51
8.15	4.80	2.28	7.20	4.80	3.96
6.90	5.70	3.90	6.50	6.50	4.55
5.75	7.15	3.41	5.40	7.90	4.68
4.90	8.90	-0.18	4.50	8.10	0.79
4.00	7.55	0.40	3.55	6.90	-3.50
4.10	6.00	-2.34	4.30	5.15	-1.48
4.25	4.40	-2.22	4.10	3.20	-2.55
4.67	2.40	-0.24	3.90	1.90	0.27
3.00	1.60	-0.50	2.50	1.00	-0.71
$\sum$		11.43	$\sum$		12.21
TOTAL		**23.64**			

Table 7: Surface area (unit^2) of Britain calculated from Green's theorem and the respective coordinates of **Figure 10**. Sample calculation is shown for the first row.

To account for the scale factor of 100 km, each of the coordinates in **Table 7** must be multiplied by 100. When the sum of the surface area is now taken, one should attain a value of 236,443 km^2 (i.e the surface area of Britain).

Since the surface area is now know, one can solve for (29):

$$D = \frac{2\log\left(\frac{P_2 G_1}{P_1 G_2}\right)}{\log\left(\frac{A_2 (G_1)^2}{A_1 (G_2)^2}\right)} \tag{37}$$

$$D = \frac{2\log\left(\frac{4{,}250*100}{3{,}300*50}\right)}{\log\left(\frac{236{,}443(100)^2}{236{,}443(50)^2}\right)} \tag{38}$$

$$D = 1.365 \tag{39}$$

Again, the percentage error between the fractal dimension determined via the relationship between the surface area and perimeter of a fractal structure (i.e. coastline of Britain) and Mandelbrot's proposed value for the fractal dimension of Britain will be determined:

$$\delta = \left|\frac{(V_A) - (V_E)}{V_E}\right| 100$$

$$\delta = \left|\frac{(1.365) - (1.25)}{1.25}\right| 100$$

$$\delta = 9.2\% \tag{40}$$

4 Determining the Fractal Dimension of Morocco

4.1 Percentage Error Analysis

	Hausdorff Method	Minkowski Bouligand Method	Area & Perimeter Method
Percentage Error (%)	1.336	7.112	9.2

Table 8: Percentage error associated with each method used to determine fractal dimension

After assessing the percentage error values of the three methods, it is evident that the Hausdorff method deviates the least from the literature value of $D = 1.25$, with a percentage error of 1.336. Given the fact that this method is the most accurate compared to the other two, the Hausdorff method will be utilized to determine the fractal dimension of Morocco.

4.2 Hausdorff Method to Determine Fractal Dimension of Morocco

To visualize the Hausdorff method, I used Photoshop to apply different sized rulers around the coast of Morocco, as shown in the following figures.

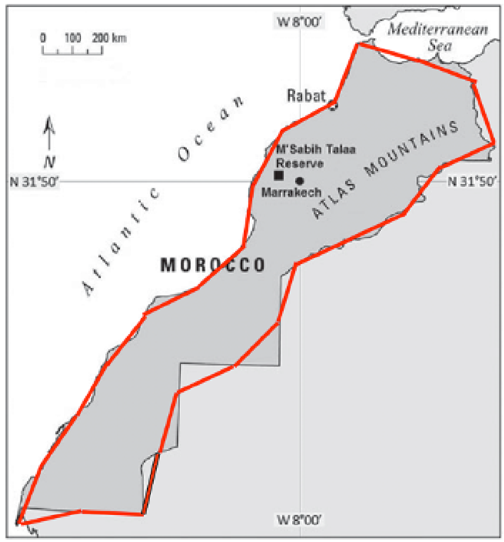

Figure 11: Modeling Moroccan coastline with 200 km ruler. The original image of the map was taken from Research Gate [11]

	Length of Ruler (G) (km)	$\log G$	Number of Rulers (n)	Length of Coastline (L)	$\log L$
Values	200	2.30	25	5000	3.70

Table 9: Properties of **Figure 10** needed to determine fractal dimension (D)

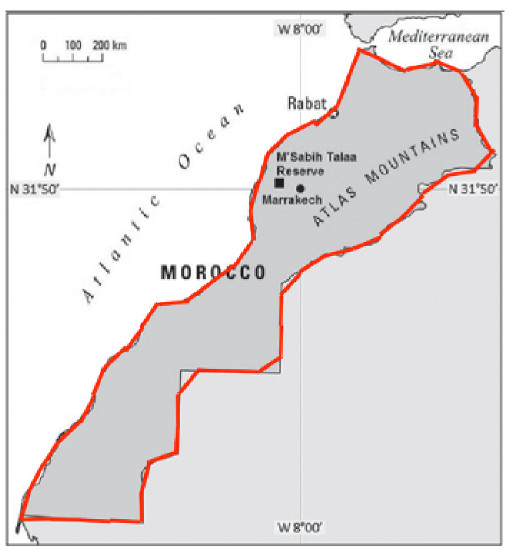

Figure 12: Modeling Moroccan coastline with 100 km ruler

	Length of Ruler (G) (km)	$\log G$	Number of Rulers (n)	Length of Coastline (L)	$\log L$
Values	100	2.00	52	5200	3.72

Table 10: Properties of **Figure 11** needed to determine fractal dimension (D)

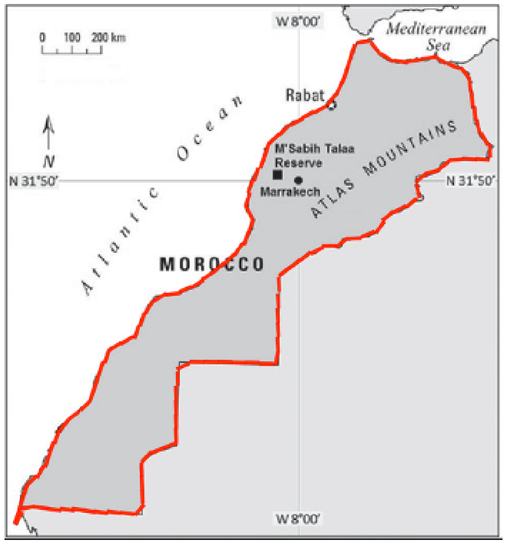

Figure 13: Modeling Moroccan coastline with 50 km ruler

	Length of Ruler (G) (km)	$\log G$	Number of Rulers (n)	Length of Coastline (L)	$\log L$
Values	50	1.70	115	5750	3.76

Table 11: Properties of **Figure 12** needed to determine fractal dimension (D)

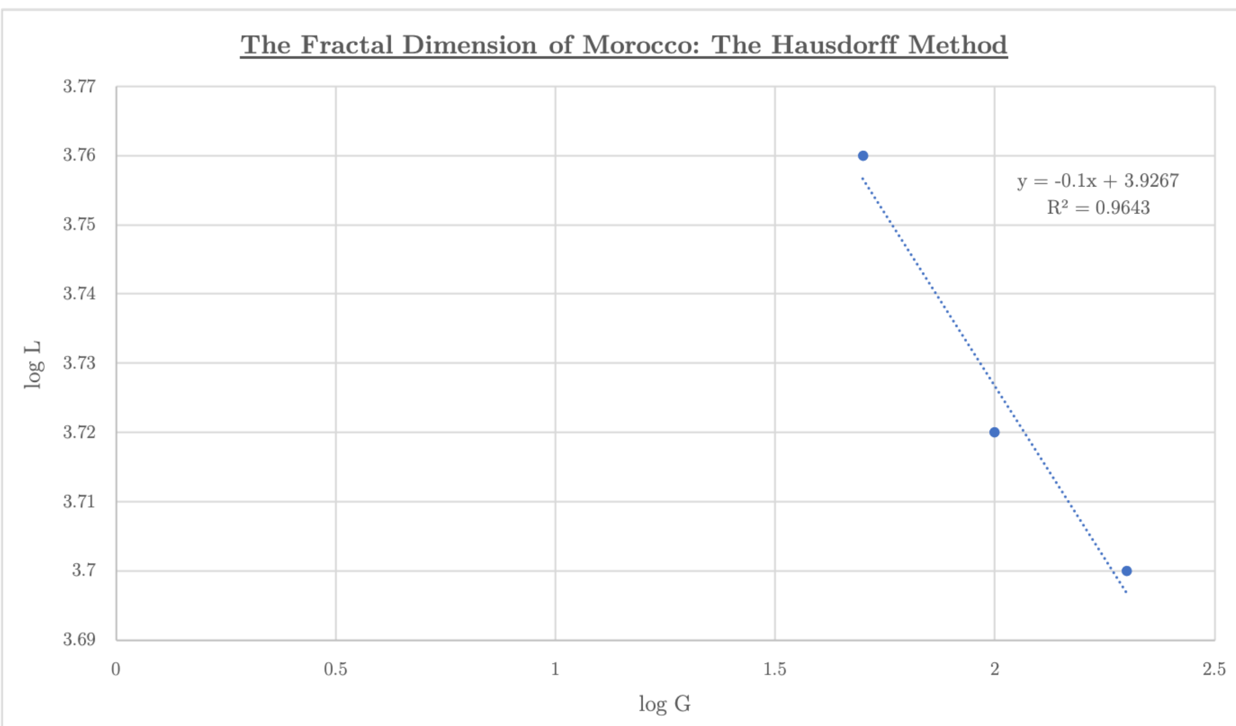

Figure 14: Plot of $\log G$ and $\log L$ from **Tables 9-11** to determine fractal dimension of Morocco

As shown in (12), the fractal dimension can be determined from the slope of **Figure 14**:

$$1 - D = m$$

$$D = 1 - m$$

$$D = 1 - (-0.1)$$

$$D = 1.1$$

Since a literature value for the fractal dimension of Morocco is not present, the percentage error value acquired from the Hausdorff method when applied to Britain (see **3.1**) will be utilized instead.

	Percentage Error (%)	Absolute Error	Maximum Value	Minimum Value
Morocco	1.336	0.012	1.112	1.088

Table 12: Percentage error analysis of fractal dimension of Morocco from Hausdorff method

5 Discussion

5.1 Conclusion & Evaluation

Although I am still unable to answer my father's initial question ("Where does this beach end?"), I have been able to answer my own ("how can I most meaningfully, yet accurately, measure the coastline of Morocco?"). By testing the accuracy of the Hausdorff method, the Minkowski Bouligand method, and the area and perimeter method against the literature value for Britain's fractal dimension, I have determined the Hausdorff method to be the most meaningfully accurate method of determining the fractal dimension of Morocco (at least from the methods explored in this paper). By using the Hausdorff method, I have also determined that Morocco possesses a fractal dimension of 1.1 with a percentage error of 1.336%.

Although one may assume that this investigation was successful in achieving its aims, there are still several limitations that must be discussed. Firstly, although the Hausdorff method may have been the most accurate method in this investigation, it still possesses certain flaws. For instance, the subjectivity associated with placing rulers around the perimeter of a map will have likely contributed to a bias in the subsequent calculations to determine the fractal dimension. In order to reduce the severity of this inherent flaw, I should have had several participants place the rulers around the perimeter of the map and take the average of the values that they have obtained. By doing this, I will be able to reduce and 'diversify' the bias that may be created from a single person conducting the Hausdorff method, possibly allowing for a more accurate fractal dimension value being obtained. This same suggestion should also be applied to the Minkowski Bouligand method as it too possesses the possibility to produce biased results due to the inherent subjectivity present when conducting the method. Interestingly, although the area and perimeter method was the most mathematically complex method present in this paper, it possessed the highest percentage error value. The obvious culprit would be Green's theorem. Just like the Hausdorff method, placing the coordinates around the map of Britain is quite subjective, hence creating a bias in the subsequent calculations to determine the fractal dimension. Furthermore, Green's theorem seems to be most effective when used in well-defined functions rather than crude objects such as **Figure 10**. One may suggest that utilizing a literature value for the surface area of Britain may have yielded a more objective and accurate value for the fractal dimensions. Interestingly, this isn't the case. Although the surface area of Britain (236,433 km^2) that was obtained through Green's theorem is nearly two-fold the literature value (130,395 km^2), they both yield the same fractal dimension:

$$D = \frac{2\log\left(\frac{P_2 G_1}{P_1 G_2}\right)}{\log\left(\frac{A_2 (G_1)^2}{A_1 (G_2)^2}\right)} \tag{41}$$

$$D = \frac{2\log\left(\frac{4{,}250*100}{3{,}300*50}\right)}{\log\left(\frac{130{,}395(100)^2}{130{,}395(50)^2}\right)} \tag{42}$$

$$D = 1.365 \tag{43}$$

Also, there are inherent flaws with the utilization of the literature value for the surface area of Britain as it assumes that the Earth is flat and ignores any curvature present, thus yielding a smaller surface area.

Diverging from any shortcomings, there are several lessons that I have learned from this investigation, one of which is the idea of 'meaningful precision'. During my IBDP Biology and Chemistry classes, I, without much thought, always associated more precise data with quality data. However, this investigation has revealed that this is not the case; precision is only useful when it is 'meaningful'. Sure, I can measure the coastline of Morocco with a 50 cm stick, but does this measurement really hold any value or meaning? Other than the idea of 'meaningful precision', this investigation has also revealed the dichotomous relationship between scientists and mathematicians; the former actively seeks precision while the latter seems apathetic towards this cause.

As an individual aspiring to work in the healthcare field, I was quite surprised to discover the medicinal applications of fractal dimensions. Research conducted by Glenny et al. reveals how fractal dimensions have been used to model the "heterogeneities of pulmonary and myocardial flows," thus possibly allowing for significant physiology analysis at minute resolution scales, although much research remains to be done [15]. From determining the coastline of Morocco to groundbreaking medical discoveries, who would have guessed that a random question that my father chattered would hold such far-reaching weight?

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